INTERNAL FORCES IN CURVED GRID MEMBER

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ABSTRACT: *Fixed end reactions of an element are one of the requirements for the analysis of discrete structural members. The fixed end reactions or the member end actions are used to determine internal forces developed in the member. Determination of internal forces of a straight member is easy, but for the beam curved in plan, the process of finding the internal forces is very cumbersome. A comprehensive C++ program has been developed to compute fixed end reactions and internal forces for a beam curved in plan, which accommodate most of types of load case and its combinations. The results obtained from the program are represented in the graphical form i.e. twisting moment diagram, bending moment diagram and shear force diagram. The results obtained, for each load case, are validated through application of analysis software. The developed formulation is useful for grid analysis also.*

Keywords: Beam Curved In Plan, Fixed End Reactions, Internal Forces, Shear Force Diagram, Bending Moment Diagram, Twisting Moment Diagram.

INTRODUCTION:

A beam curved in plan is a plane structure assumed to be lying in the horizontal plane and all the forces are normal to the plane i.e. acting in the vertical direction. All the moments and couples have their vectors in the plane of the grid. This orientation of loadings, results in twisting moment, bending moment and shear force in the member [2]. Member end reactions or internal forces at a section are twisting moment, bending moment and shear force. For the analysis of structures by classical methods like slope deflection method, moment distribution method, Kani's method and matrix method of analysis (flexibility method and stiffness method), fixed end reactions of individual members are required. Proper representation of fixed end actions in matrix method of analysis is essential for computer programming. To find the fixed end reactions for curved girder using classical approach is very cumbersome. Equivalent joint loads are needed for the analysis of the structures, either by classical method or matrix method [1, 4, 5]. Hansora et al. (2010) [3] have formulated equations in matrices form to evaluate fixed end actions for a beam curved in plan, subjected to different types of loads. The internal forces i.e twisting moment, bending moment and shear force at any section of a beam curved in plan can be obtained using fixed end reactions. Most of the professional analysis software does not provide graphical representation of the results and information about important parameters like point of contraflexure, point of maximum moment etc. for the beam curved in plan. Here, effort is made to represent the results of internal forces obtained from C++ program graphically for the beam curved in plan. When a member is subjected to twisting moment, bending moment and vertical load, the strain energy of the member is given by following well-known equations [5, 7].

$$\int \frac{M_x}{EI}\frac{\partial M_x}{\partial T_k}ds + \int \frac{T_x}{GJ}\frac{\partial T_x}{\partial T_k}ds = 0$$

$$\int \frac{M_x}{EI}\frac{\partial M_x}{\partial M_k}ds + \int \frac{T_x}{GJ}\frac{\partial T_x}{\partial M_k}ds = 0$$

$$\int \frac{M_x}{EI}\frac{\partial M_x}{\partial V_k}ds + \int \frac{T_x}{GJ}\frac{\partial T_x}{\partial V_k}ds = 0$$

FORMULATION:

Fixed end reactions are essential for evaluation of internal forces in a beam curved in plan [3]. The formulation of the equations are based on the positive axis as shown in Fig.1, in which x-y is the horizontal plane and z is vertically upward directions. The loading cases included are twisting moment, bending moment, concentrated load and full/partial uniformly varying load (refer Fig. 1 for loading notations).

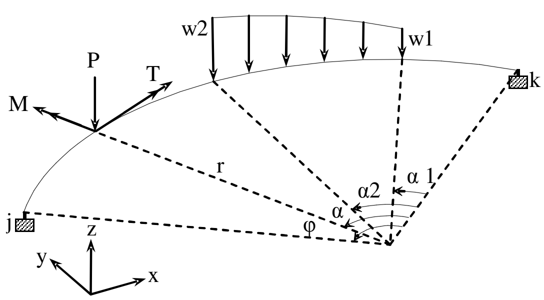


Fig. 1: Positive Axis and Loading Notations

To find the fixed end reactions in local axis, at k-end and j-end, the following equations are to be used.

$$[GR]_k \{AR\}_k + \{GL\}_k = 0$$

The above equation can be re-written as

$$\{AR\}_k = -[GR]_k^{-1}\{GL\}_k$$

$$\{AR\}_j = [GR]_j \{AR\}_k + \{GL\}_j$$

Full formulation of above matrices is as follows:

$$[GR]_k = \begin{bmatrix} GR_{11} & GR_{12} & GR_{13} \\ GR_{21} & GR_{22} & GR_{23} \\ GR_{31} & GR_{32} & GR_{33} \end{bmatrix}$$

$$GR_{11} = r\left[\frac{1}{EI}\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right) + \frac{1}{GJ}\left(\frac{\phi}{2} + \frac{\sin 2\phi}{4}\right)\right]$$

$$GR_{12} = r\left[\frac{1}{EI}\left(-\frac{\sin^2\phi}{2}\right) + \frac{1}{GJ}\left(\frac{\sin^2\phi}{2}\right)\right]$$

$$GR_{13} = r^2\left[\frac{1}{EI}\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right) + \frac{1}{GJ}\left(\frac{\phi}{2} + \frac{\sin 2\phi}{4} - \sin\phi\right)\right]$$

$$GR_{21} = GR_{12}$$

$$GR_{22} = r\left[\frac{1}{EI}\left(\frac{\phi}{2} + \frac{\sin 2\phi}{4}\right) + \frac{1}{GJ}\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)\right]$$

$$GR_{23} = r^2\left[\frac{1}{EI}\left(-\frac{\sin^2\phi}{2}\right) - \frac{1}{GJ}\left(1 - \frac{\sin^2\phi}{2} - \cos\phi\right)\right]$$

$$GR_{31} = GR_{13} \qquad GR_{32} = GR_{23}$$

$$GR_{33} = r^3\left[\frac{1}{EI}\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right) + \frac{1}{GJ}\left(\frac{3\phi}{2} + \frac{\sin 2\phi}{4} - 2\sin\phi\right)\right]$$

$$\{GL\}_k = \begin{Bmatrix} GL_{11} \\ GL_{21} \\ GL_{31} \end{Bmatrix} \quad \{AR\}_k = \begin{Bmatrix} T_k \\ M_k \\ V_k \end{Bmatrix} \quad \{AR\}_j = \begin{Bmatrix} T_j \\ M_j \\ V_j \end{Bmatrix}$$

$$[GR]_j = \begin{bmatrix} -\cos\phi & -\sin\phi & r(1-\cos\phi) \\ \sin\phi & -\cos\phi & r\sin\phi \\ 0 & 0 & -1 \end{bmatrix}$$

Constants for Load Case – 1, 2 & 3

$$C_1 = \left(\frac{(\phi-\alpha)}{2} + \frac{\sin 2\phi}{4} - \frac{\sin 2\alpha}{4}\right)$$

$$C_2 = \left(\frac{(\phi-\alpha)}{2} - \frac{\sin 2\phi}{4} + \frac{\sin 2\alpha}{4}\right)$$

$$C_3 = \left(-\frac{\cos^2\phi}{2} + \frac{\cos^2\alpha}{2}\right)$$

Load Case – 1: Twisting Moment

$$GL_{11} = Tr\left\{\begin{array}{l} \frac{1}{EI}\left[\cos\alpha\, C_2 - \sin\alpha\, C_3\right] \\ + \frac{1}{GJ}\left[\cos\alpha\, C_1 + \sin\alpha\, C_3\right] \end{array}\right\}$$

$$GL_{21} = Tr\left\{\begin{array}{l} \frac{1}{EI}\left[-\cos\alpha\, C_3 + \sin\alpha\, C_1\right] \\ + \frac{1}{GJ}\left[\cos\alpha\, C_3 + \sin\alpha\, C_2\right] \end{array}\right\}$$

$$GL_{31} = Tr^2\left\{\begin{array}{l} \frac{1}{EI}\left[\cos\alpha\, C_2 - \sin\alpha\, C_3\right] \\ + \frac{1}{GJ}\left[-\sin(\phi-\alpha) + \cos\alpha\, C_1 + \sin\alpha\, C_3\right] \end{array}\right\}$$

$$\{GL\}_j = -T\begin{Bmatrix} \cos(\phi-\alpha) \\ -\sin(\phi-\alpha) \\ 0 \end{Bmatrix}$$

Load Case – 2: Bending Moment

$$GL_{11} = Mr\left\{\begin{array}{l} \frac{1}{EI}\left[-\cos\alpha\, C_3 - \sin\alpha\, C_2\right] \\ + \frac{1}{GJ}\left[\cos\alpha\, C_3 - \sin\alpha\, C_1\right] \end{array}\right\}$$

$$GL_{21} = Mr\left\{\begin{array}{l} \frac{1}{EI}\left[\cos\alpha\, C_1 + \sin\alpha\, C_3\right] \\ + \frac{1}{GJ}\left[\cos\alpha\, C_2 - \sin\alpha\, C_3\right] \end{array}\right\}$$

$$GL_{31} = Mr^2\left\{\begin{array}{l} \frac{1}{EI}\left[-\cos\alpha\, C_3 - \sin\alpha\, C_2\right] \\ + \frac{1}{GJ}\left[\cos(\phi-\alpha) - 1 + \cos\alpha\, C_3 - \sin\alpha\, C_1\right] \end{array}\right\}$$

$$\{GL\}_j = -M\begin{Bmatrix} \sin(\phi-\alpha) \\ \cos(\phi-\alpha) \\ 0 \end{Bmatrix}$$

Load Case – 3: Concentrated Load (Fig. 3.4)

$$GL_{11} = Pr^2\left\{\begin{array}{l} \frac{1}{EI}\left[-\cos\alpha\, C_2 + \sin\alpha\, C_3\right] \\ + \frac{1}{GJ}\left[(\sin\phi - \sin\alpha) - \cos\alpha\, C_1 - \sin\alpha\, C_3\right] \end{array}\right\}$$

$$GL_{21} = Pr^2\left\{\begin{array}{l} \frac{1}{EI}\left[\cos\alpha\, C_3 - \sin\alpha\, C_1\right] \\ + \frac{1}{GJ}\left[(-\cos\phi + \cos\alpha) - \cos\alpha\, C_3 - \sin\alpha\, C_2\right] \end{array}\right\}$$

$$GL_{31} = Pr^3\left\{\begin{array}{l} \frac{1}{EI}\left[-\cos\alpha\, C_2 + \sin\alpha\, C_3\right] \\ + \frac{1}{GJ}\left[\begin{array}{l} -(\phi-\alpha) + (\sin\phi - \sin\alpha) \\ + \sin(\phi-\alpha) - \cos\alpha\, C_1 - \sin\alpha\, C_3 \end{array}\right] \end{array}\right\}$$

$$\{GL\}_j = -Pr\begin{Bmatrix} 1 - \cos(\phi-\alpha) \\ \sin(\phi-\alpha) \\ -\frac{1}{r} \end{Bmatrix}$$

Load Case – 4: Partial Uniformly Varying Load

$$P_1 = w_1 - P_2\alpha_1$$

$$P_2 = \frac{(w_2 - w_1)}{(\alpha_2 - \alpha_1)}$$

$$P_3 = (\alpha_2 - \alpha_1)\left[P_1 + \frac{P_2(\alpha_2 + \alpha_1)}{2}\right]$$

$$P_4 = -\left(P_1\alpha_1 + P_2 + \frac{P_2\alpha_1^2}{2}\right)$$

$$P_5 = -\left(P_1\alpha_2 + P_2 + \frac{P_2\alpha_2^2}{2}\right)$$

$$C_1 = \left(\frac{(\phi - \alpha_1)}{2} + \frac{\sin 2\phi}{4} - \frac{\sin 2\alpha_1}{4}\right)$$

$$C_2 = \left(\frac{(\phi - \alpha_1)}{2} - \frac{\sin 2\phi}{4} + \frac{\sin 2\alpha_1}{4}\right)$$

$$C_3 = \left(-\frac{\cos^2\phi}{2} + \frac{\cos^2\alpha_1}{2}\right)$$

$$C_4 = \left(\frac{(\phi - \alpha_2)}{2} + \frac{\sin 2\phi}{4} - \frac{\sin 2\alpha_2}{4}\right)$$

$$C_5 = \left(\frac{(\phi - \alpha_2)}{2} - \frac{\sin 2\phi}{4} + \frac{\sin 2\alpha_2}{4}\right)$$

$$C_6 = \left(-\frac{\cos^2\phi}{2} + \frac{\cos^2\alpha_2}{2}\right)$$

$$GL1_{11} = r^3\left\{\begin{array}{l} \dfrac{1}{EI}\left[\begin{array}{l} P_1(\cos\phi - \cos\alpha_1) \\ -P_2(\sin\phi - \sin\alpha_1 - \phi\cos\phi + \alpha_1\cos\alpha_1) \\ +w_1\left[\cos\alpha_1\, C_3 + \sin\alpha_1\, C_2\right] \\ +P_2\left[\cos\alpha_1\, C_2 - \sin\alpha_1\, C_3\right] \end{array}\right] \\ +\dfrac{1}{GJ}\left[\begin{array}{l} P_4(\sin\phi - \sin\alpha_1) \\ +P_1(\cos\phi - \cos\alpha_1 + \phi\sin\phi - \alpha_1\sin\alpha_1) \\ +P_2(\phi\cos\phi - \alpha_1\cos\alpha_1 + \dfrac{\phi^2}{2}\sin\phi \\ -\dfrac{\alpha_1^2}{2}\sin\alpha_1 - \sin\phi + \sin\alpha_1) \\ -w_1\left[\cos\alpha_1\, C_3 - \sin\alpha_1\, C_1\right] \\ +P_2\left[\cos\alpha_1\, C_1 + \sin\alpha_1\, C_3\right] \end{array}\right] \end{array}\right\}$$

$$GL2_{11} = r^3\left\{\begin{array}{l} \dfrac{1}{EI}\left[\begin{array}{l} P_1(\cos\phi - \cos\alpha_2) \\ -P_2(\sin\phi - \sin\alpha_2 - \phi\cos\phi + \alpha_2\cos\alpha_2) \\ +w_2\left[\cos\alpha_2\, C_6 + \sin\alpha_2\, C_5\right] \\ +P_2\left[\cos\alpha_2\, C_5 - \sin\alpha_2\, C_6\right] \end{array}\right] \\ +\dfrac{1}{GJ}\left[\begin{array}{l} P_5(\sin\phi - \sin\alpha_2) \\ +P_1(\cos\phi - \cos\alpha_2 + \phi\sin\phi - \alpha_2\sin\alpha_2) \\ +P_2(\phi\cos\phi - \alpha_2\cos\alpha_2 + \dfrac{\phi^2}{2}\sin\phi \\ -\dfrac{\alpha_2^2}{2}\sin\alpha_2 - \sin\phi + \sin\alpha_2) \\ -w_2\left[\cos\alpha_2\, C_6 - \sin\alpha_2\, C_4\right] \\ +P_2\left[\cos\alpha_2\, C_4 + \sin\alpha_2\, C_6\right] \end{array}\right] \end{array}\right\}$$

$$GL1_{21} = r^3\left\{\begin{array}{l} \dfrac{1}{EI}\left[\begin{array}{l} P_1(\sin\phi - \sin\alpha_1) \\ +P_2(\cos\phi - \cos\alpha_1 + \phi\sin\phi - \alpha_1\sin\alpha_1) \\ -w_1\left[\cos\alpha_1\, C_1 + \sin\alpha_1\, C_3\right] \\ -P_2\left[\cos\alpha_1\, C_3 - \sin\alpha_1\, C_1\right] \end{array}\right] \\ +\dfrac{1}{GJ}\left[\begin{array}{l} P_4(-\cos\phi + \cos\alpha_1) \\ +P_1(\sin\phi - \sin\alpha_1 - \phi\cos\phi + \alpha_1\cos\alpha_1) \\ +P_2(\phi\sin\phi - \alpha_1\sin\alpha_1 - \dfrac{\phi^2}{2}\cos\phi \\ +\dfrac{\alpha_1^2}{2}\cos\alpha_1 + \cos\phi - \cos\alpha_1) \\ -w_1\left[\cos\alpha_1\, C_2 - \sin\alpha_1\, C_3\right] \\ +P_2\left[\cos\alpha_1\, C_3 + \sin\alpha_1\, C_2\right] \end{array}\right] \end{array}\right\}$$

$$GL2_{21} = r^3\left\{\begin{array}{l} \dfrac{1}{EI}\left[\begin{array}{l} P_1(\sin\phi - \sin\alpha_2) \\ +P_2(\cos\phi - \cos\alpha_2 + \phi\sin\phi - \alpha_2\sin\alpha_2) \\ -w_2\left[\cos\alpha_2\, C_4 + \sin\alpha_2\, C_6\right] \\ -P_2\left[\cos\alpha_2\, C_6 - \sin\alpha_2\, C_4\right] \end{array}\right] \\ +\dfrac{1}{GJ}\left[\begin{array}{l} P_5(-\cos\phi + \cos\alpha_2) \\ +P_1(\sin\phi - \sin\alpha_2 - \phi\cos\phi + \alpha_2\cos\alpha_2) \\ +P_2(\phi\sin\phi - \alpha_2\sin\alpha_2 - \dfrac{\phi^2}{2}\cos\phi \\ +\dfrac{\alpha_2^2}{2}\cos\alpha_2 + \cos\phi - \cos\alpha_2) \\ -w_2\left[\cos\alpha_2\, C_5 - \sin\alpha_2\, C_6\right] \\ +P_2\left[\cos\alpha_2\, C_6 + \sin\alpha_2\, C_5\right] \end{array}\right] \end{array}\right\}$$

$$
GL1_{31} = r^4 \left\{ \begin{array}{l} \dfrac{1}{EI} \left[\begin{array}{l} P_1(\cos\phi - \cos\alpha_1) \\ -P_2(\sin\phi - \sin\alpha_1 - \phi\cos\phi + \alpha_1\cos\alpha_1) \\ +w_1\left[\cos\alpha_1 C_3 + \sin\alpha_1 C_2\right] \\ +P_2\left[\cos\alpha_1 C_2 - \sin\alpha_1 C_3\right] \end{array} \right] \\ +\dfrac{1}{GJ} \left[\begin{array}{l} -P_4(\phi - \alpha_1) - P_1\left(\dfrac{\phi^2 - \alpha_1^2}{2}\right) \\ -P_2\left(\dfrac{(\phi^3 - \alpha_1^3)}{6} + \sin(\phi - \alpha_1)\right) \\ +w_1\left[-\cos(\phi - \alpha_1) + 1\right] + P_4(\sin\phi - \sin\alpha_1) \\ +P_1(\cos\phi - \cos\alpha_1 + \phi\sin\phi - \alpha_1\sin\alpha_1) \\ +P_2(\phi\cos\phi - \alpha_1\cos\alpha_1 + \dfrac{\phi^2}{2}\sin\phi \\ -\dfrac{\alpha_1^2}{2}\sin\alpha_1 - \sin\phi + \sin\alpha_1) \\ -w_1\left[\cos\alpha_1 C_3 - \sin\alpha_1 C_1\right] \\ +P_2\left[\cos\alpha_1 C_1 + \sin\alpha_1 C_3\right] \end{array} \right] \end{array} \right\}
$$

$$
GL2_{31} = r^4 \left\{ \begin{array}{l} \dfrac{1}{EI} \left[\begin{array}{l} P_1(\cos\phi - \cos\alpha_2) \\ -P_2(\sin\phi - \sin\alpha_2 - \phi\cos\phi + \alpha_2\cos\alpha_2) \\ +w_2\left[\cos\alpha_2 C_6 + \sin\alpha_2 C_5\right] \\ +P_2\left[\cos\alpha_2 C_5 - \sin\alpha_2 C_6\right] \end{array} \right] \\ +\dfrac{1}{GJ} \left[\begin{array}{l} -P_5(\phi - \alpha_2) - P_1\left(\dfrac{\phi^2 - \alpha_2^2}{2}\right) \\ -P_2\left(\dfrac{(\phi^3 - \alpha_2^3)}{6} + \sin(\phi - \alpha_2)\right) \\ +w_2\left[-\cos(\phi - \alpha_2) + 1\right] \\ +P_5(\sin\phi - \sin\alpha_2) \\ +P_1(\cos\phi - \cos\alpha_2 + \phi\sin\phi - \alpha_2\sin\alpha_2) \\ +P_2(\phi\cos\phi - \alpha_2\cos\alpha_2 + \dfrac{\phi^2}{2}\sin\phi \\ -\dfrac{\alpha_1^2}{2}\sin\alpha_2 - \sin\phi + \sin\alpha_2) \\ -w_2\left[\cos\alpha_2 C_6 - \sin\alpha_2 C_4\right] \\ +P_2\left[\cos\alpha_2 C_4 + \sin\alpha_2 C_6\right] \end{array} \right] \end{array} \right\}
$$

$$GL_{11} = GL1_{11} - GL2_{11}$$
$$GL_{21} = GL1_{21} - GL2_{21}$$
$$GL_{31} = GL1_{31} - GL2_{31}$$

$$
\{GL\}_j = -r^2 \left\{ \begin{array}{c} \left\{ \begin{array}{l} P_4 - P_5 - w_1\sin(\phi - \alpha_1) + w_2\sin(\phi - \alpha_2) \\ +P_2\left[\cos(\phi - \alpha_1) - \cos(\phi - \alpha_2)\right] \end{array} \right\} \\ \left\{ \begin{array}{l} -w_1\cos(\phi - \alpha_1) + w_2\cos(\phi - \alpha_2) \\ -P_2\left[\sin(\phi - \alpha_1) - \sin(\phi - \alpha_2)\right] \end{array} \right\} \\ \left\{ -\dfrac{1}{r}\left[(w_1 + w_2)\left(\dfrac{(\alpha_2 - \alpha_1)}{2}\right)\right] \right\} \end{array} \right\}
$$

Now, the equations of internal forces for the various load cases are as follows:

Load Case – 1: Twisting Moment

When $\theta = 0$

$T_{xR} = 0$

$T_{xL} = T_k\cos\theta + M_k\sin\theta - V_k r(1 - \cos\theta)$

$M_{xR} = 0$

$M_{xL} = -T_k\sin\theta + M_k\cos\theta - V_k r\sin\theta$

$V_{xR} = 0$ $\quad$ $V_{xL} = V_k$

When $0 < \theta < \alpha$

$T_{xR} = T_k\cos\theta + M_k\sin\theta - V_k r(1 - \cos\theta)$

$T_{xL} = T_k\cos\theta + M_k\sin\theta - V_k r(1 - \cos\theta)$

$M_{xR} = -T_k\sin\theta + M_k\cos\theta - V_k r\sin\theta$

$M_{xL} = -T_k\sin\theta + M_k\cos\theta - V_k r\sin\theta$

$V_{xR} = V_k$ $\quad$ $V_{xL} = V_k$

When $\theta = \alpha$

$T_{xR} = T_k\cos\theta + M_k\sin\theta - V_k r(1 - \cos\theta)$

$T_{xL} = T_k\cos\theta + M_k\sin\theta - V_k r(1 - \cos\theta) + T$

$M_{xR} = -T_k\sin\theta + M_k\cos\theta - V_k r\sin\theta$

$M_{xL} = -T_k\sin\theta + M_k\cos\theta - V_k r\sin\theta$

$V_{xR} = V_k$ $\quad$ $V_{xL} = V_k$

When $\alpha < \theta < \phi$

$T_{xR} = T_k\cos\theta + M_k\sin\theta - V_k r(1 - \cos\theta) + T(\theta - \alpha)$

$T_{xL} = T_k\cos\theta + M_k\sin\theta - V_k r(1 - \cos\theta) + T(\theta - \alpha)$

$M_{xR} = -T_k\sin\theta + M_k\cos\theta - V_k r(1 - \cos\theta) - T\sin(\theta - \alpha)$

$M_{xL} = -T_k\sin\theta + M_k\cos\theta - V_k r(1 - \cos\theta) - T\sin(\theta - \alpha)$

$V_{xR} = V_k$ $\quad$ $V_{xL} = V_k$

When $\theta = \phi$

$T_{xR} = T_k\cos\theta + M_k\sin\theta - V_k r(1 - \cos\theta) + T(\theta - \alpha)$

$T_{xL} = 0$

$M_{xR} = -T_k\sin\theta + M_k\cos\theta - V_k r(1 - \cos\theta) - T\sin(\theta - \alpha)$

$M_{xL} = 0$

$V_{xR} = V_k$ $\quad$ $V_{xL} = 0$

Load Case – 2: Bending Moment

When $\theta = 0$

$T_{xR} = 0$

$T_{xL} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$M_{xR} = 0$

$M_{xL} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$

$V_{xR} = 0 \qquad V_{xL} = V_k$

When $0 < \theta < \alpha$

$T_{xR} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$T_{xL} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$M_{xR} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$

$M_{xL} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$

$V_{xR} = V_k \qquad V_{xL} = V_k$

When $\theta = \alpha$

$T_{xR} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$T_{xL} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$M_{xR} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$

$M_{xL} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta + M$

$V_{xR} = V_k \qquad V_{xL} = V_k$

When $\alpha < \theta < \phi$

$T_{xR} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta) + M \sin(\theta-\alpha)$

$T_{xL} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta) + M \sin(\theta-\alpha)$

$M_{xR} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta - M\cos(\theta-\alpha)$

$M_{xL} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta - M\cos(\theta-\alpha)$

$V_{xR} = V_k \qquad V_{xL} = V_k$

When $\theta = \phi$

$T_{xR} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta) + M \sin(\theta-\alpha)$

$T_{xL} = 0$

$M_{xR} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta - M\cos(\theta-\alpha)$

$M_{xL} = 0$

$V_{xR} = V_k \qquad V_{xL} = 0$

Load Case – 3: Concentrated Load

When $\theta = 0$

$T_{xR} = 0$

$T_{xL} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$M_{xR} = 0$

$M_{xL} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$

$V_{xR} = 0 \qquad V_{xL} = V_k$

When $0 < \theta < \alpha$

$T_{xR} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$T_{xL} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$M_{xR} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$

$M_{xL} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$

$V_{xR} = V_k \qquad V_{xL} = V_k$

When $\theta = \alpha$

$T_{xR} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$T_{xL} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$M_{xR} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$

$M_{xL} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$

$V_{xR} = V_k \qquad V_{xL} = V_k - P$

When $\alpha < \theta < \phi$

$T_{xR} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$
$+ P\, r\, (1-\cos(\theta-\alpha))$

$T_{xL} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$
$+ P\, r\, (1-\cos(\theta-\alpha))$

$M_{xR} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$
$- P\, r\, (1-\sin(\theta-\alpha))$

$M_{xL} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$
$- P\, r\, (1-\sin(\theta-\alpha))$

$V_{xR} = V_k - P \qquad V_{xL} = V_k - P$

When $\theta = \phi$

$T_{xR} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$
$+ P\, r\, (1-\cos(\theta-\alpha))$

$T_{xL} = 0$

$M_{xR} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$
$- P\, r\, (1-\sin(\theta-\alpha))$

$M_{xL} = 0$

$V_{xR} = V_k - P \qquad V_{xL} = 0$

Load Case – 4: Full/Partial Uniformly Varying Load

When $\theta = 0$

$T_{xR} = 0$

$T_{xL} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$M_{xR} = 0$

$M_{xL} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$

$V_{xR} = 0 \qquad V_{xL} = V_k$

When $0 < \theta < \alpha l$

$T_{xR} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$T_{xL} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta)$

$M_{xR} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$

$M_{xL} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta$

$V_{xR} = V_k \qquad V_{xL} = V_k$

When $\alpha 1 < \theta \le \alpha 2$

$$T_{xR} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta) + r^2\begin{bmatrix} P_4 + P_1\theta + 0.5P_2\theta^2 - w_1 \sin(\theta-\alpha_1) \\ + P_2 \cos(\theta-\alpha_1) \end{bmatrix}$$

$$T_{xl} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta) + r^2\begin{bmatrix} P_4 + P_1\theta + 0.5P_2\theta^2 - w_1 \sin(\theta-\alpha_1) \\ + P_2 \cos(\theta-\alpha_1) \end{bmatrix}$$

$$M_{xR} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta + r^2\left[P_1 + P_2\theta - w_1 \cos(\theta-\alpha_1) - P_2 \sin(\theta-\alpha_1)\right]$$

$$M_{xL} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta + r^2\left[P_1 + P_2\theta - w_1 \cos(\theta-\alpha_1) - P_2 \sin(\theta-\alpha_1)\right]$$

$$V_{xR} = V_k - 0.5r\left[2w_1 + \frac{(w_2 - w_1)}{(\alpha_2 - \alpha_1)}(\theta-\alpha_1)\right](\theta-\alpha_1)$$

$$V_{xL} = V_k - 0.5r\left[2w_1 + \frac{(w_2 - w_1)}{(\alpha_2 - \alpha_1)}(\theta-\alpha_1)\right](\theta-\alpha_1)$$

When $\alpha 2 < \theta < \phi$

$$T_{xR} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta) + r^2\begin{bmatrix} P_4 + P_1\theta + 0.5P_2\theta^2 - w_1 \sin(\theta-\alpha_1) \\ + P_2 \cos(\theta-\alpha_1) \end{bmatrix} - r^2\begin{bmatrix} p_5 + P_1\theta + 0.5P_2\theta^2 - w_2 \sin(\theta-\alpha_2) \\ + P_2 \cos(\theta-\alpha_2) \end{bmatrix}$$

$$T_{xL} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta) + r^2\begin{bmatrix} P_4 + P_1\theta + 0.5P_2\theta^2 - w_1 \sin(\theta-\alpha_1) \\ + P_2 \cos(\theta-\alpha_1) \end{bmatrix} - r^2\begin{bmatrix} P_5 + P_1\theta + 0.5P_2\theta^2 - w_2 \sin(\theta-\alpha_2) \\ + P_2 \cos(\theta-\alpha_2) \end{bmatrix}$$

$$M_{xR} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta + r^2\left[P_1 + P_2\theta - w_1 \cos(\theta-\alpha_1) - P_2 \sin(\theta-\alpha_1)\right] - r^2\left[p_1 + p_2\theta - w_2 \cos(\theta-\alpha_2) - p_2 \sin(\theta-\alpha_2)\right]$$

$$M_{xL} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta + r^2\left[P_1 + P_2\theta - w_1 \cos(\theta-\alpha_1) - P_2 \sin(\theta-\alpha_1)\right] - r^2\left[P_1 + P_2\theta - w_2 \cos(\theta-\alpha_2) - P_2 \sin(\theta-\alpha_2)\right]$$

$$V_{xR} = V_k - 0.5r(w_1 + w_2)(\alpha_2 - \alpha_1)$$

$$V_{xL} = V_k - 0.5r(w_1 + w_2)(\alpha_2 - \alpha_1)$$

When $\theta = \phi$

$$T_{xR} = T_k \cos\theta + M_k \sin\theta - V_k r(1-\cos\theta) + r^2\begin{bmatrix} P_4 + P_1\theta + 0.5P_2\theta^2 - w_1 \sin(\theta-\alpha_1) \\ + P_2 \cos(\theta-\alpha_1) \end{bmatrix} - r^2\begin{bmatrix} p_5 + P_1\theta + 0.5P_2\theta^2 - w_2 \sin(\theta-\alpha_2) \\ + P_2 \cos(\theta-\alpha_2) \end{bmatrix}$$

$$T_{xL} = 0$$

$$M_{xR} = -T_k \sin\theta + M_k \cos\theta - V_k r \sin\theta + r^2\left[P_1 + P_2\theta - w_1 \cos(\theta-\alpha_1) - P_2 \sin(\theta-\alpha_1)\right] - r^2\left[p_1 + p_2\theta - w_2 \cos(\theta-\alpha_2) - p_2 \sin(\theta-\alpha_2)\right]$$

$$M_{xL} = 0$$

$$V_{xR} = V_k - 0.5r(w_1 + w_2)(\alpha_2 - \alpha_1)$$

$$V_{xL} = 0$$

VALIDATION & APPLICATIONS:

Example – I: Cross-section – 0.300 m × 0.450 m, r = 5 m, φ = 180°, T = 50kNm, α = 60°

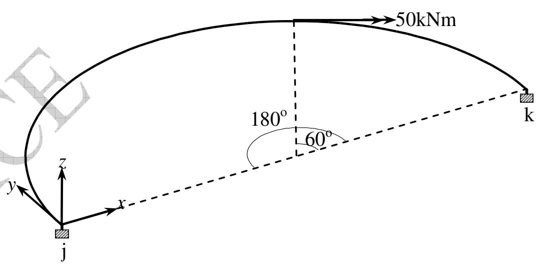
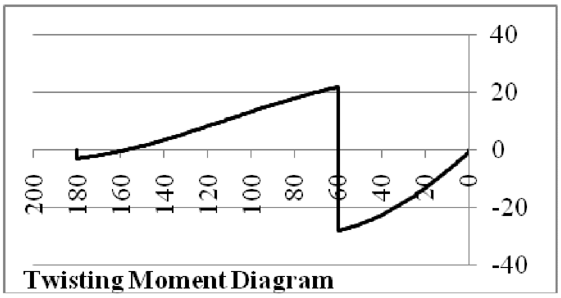


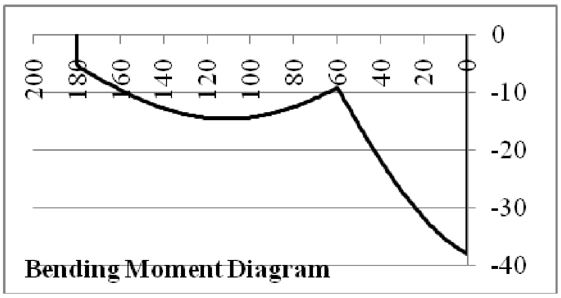
Fig. 2: Grid member subjected to twisting moment

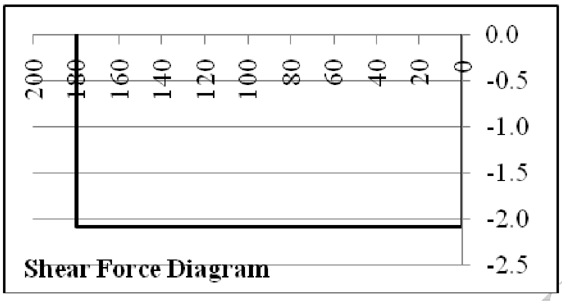
Table 1: Comparison of results of Example – I

Section		Results of the Program			Results from STAAD.Pro		
Left/ Right	θ (°)	T_x (kNm)	M_x (kNm)	V_x (kN)	T_x (kNm)	M_x (kNm)	V_x (kN)
(L)	180	0.000	0.000	0.000	0.000	0.000	0.000
(R)	180	-3.159	-5.226	-2.091	-3.159	-5.226	-2.091
	150	1.278	-11.334	-2.091	1.278	-11.334	-2.091
	120	8.174	-14.404	-2.091	8.174	-14.404	-2.091
	90	15.682	-13.615	-2.091	15.682	-13.616	-2.091
(L)	60	21.790	-9.178	-2.091	21.790	-9.178	-2.091
(R)	60	-28.210	-9.178	-2.091	-28.210	-9.178	-2.091
	30	-18.441	-27.282	-2.091	-18.441	-27.282	-2.091
(L)	0	-0.929	-38.075	-2.091	-0.929	-38.075	-2.091
(R)	0	0.000	0.000	0.000	0.000	0.000	0.000

Graphical representation of twisting moment, bending moment and shear force are as follows: [x-axis represents angle θ in degree and y-axis represents twisting moment (kNm), bending moment (kNm) and shear force (kN) according to the case]







Example – II: Cross-section – 0.300 m × 0.450 m, r = 5 m, φ = 180°, M = 50kNm, α = 60°

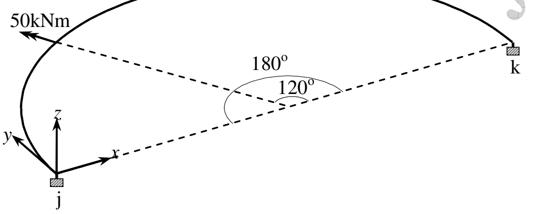
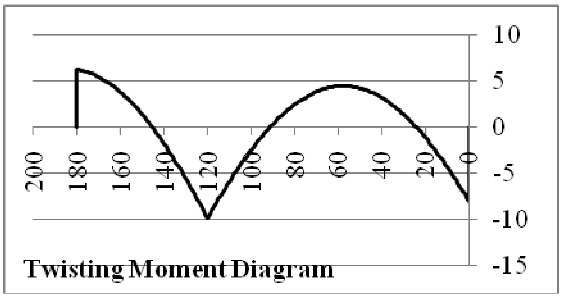
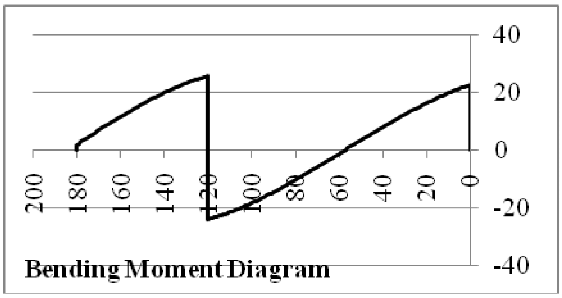


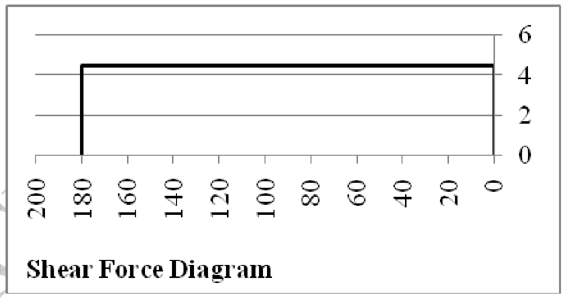
Fig. 3: Grid member subjected to bending moment

Table 2: Comparison of results of Example – II

Section		Results of the Program			Results from STAAD.Pro		
Left/ Right	θ (o)	Tx (kNm)	Mx (kNm)	Vx (kN)	Tx (kNm)	Mx (kNm)	Vx (kN)
(L)	180	0.000	0.000	0.000	0.000	0.000	0.000
(R)	180	6.336	2.108	4.506	6.336	2.108	4.506
	150	1.415	16.259	4.506	1.415	16.259	4.506
(L)	120	-9.923	26.054	4.506	-9.923	26.054	4.506
(R)	120	-9.923	-23.946	4.506	-9.923	-23.946	4.506
	90	0.361	-14.434	4.506	0.361	-14.434	4.506
	60	4.511	-1.054	4.506	4.511	-1.054	4.506
	30	1.415	12.608	4.506	1.415	12.608	4.506
(L)	0	-8.098	22.892	4.506	-8.097	22.892	4.506
(R)	0	0.000	0.000	0.000	0.000	0.000	0.000







Example – III: Cross-section – 0.300 m × 0.450 m, r = 5 m, φ = 180°, P = 50kNm, α = 120°

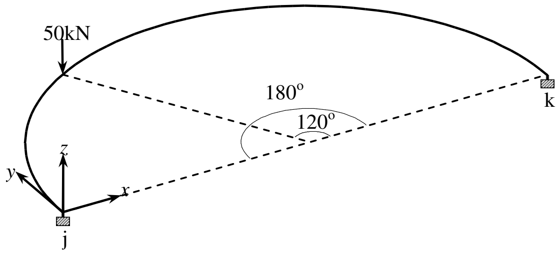
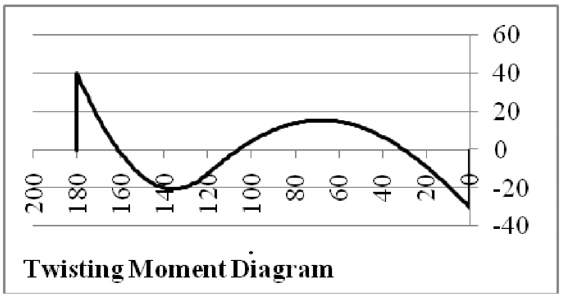


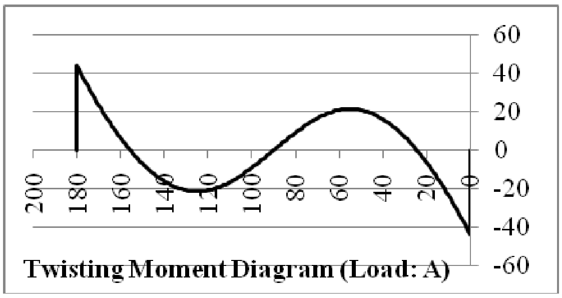
Fig. 4: Grid member subjected to concentrated load

Table 3: Comparison of results of Example – III

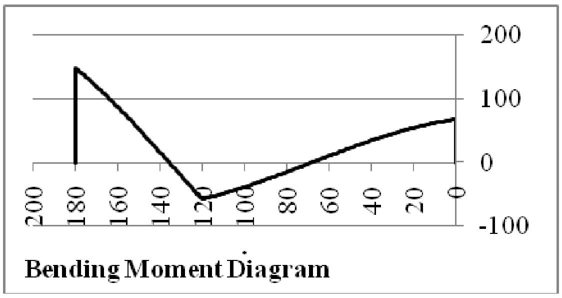
Section		Results of the Program			Results from STAAD.Pro		
Left/ Right	θ (°)	T_x (kNm)	M_x (kNm)	V_x (kN)	T_x (kNm)	M_x (kNm)	V_x (kN)
(L)	180	0.000	0.000	0.000	0.000	0.000	0.000
(R)	180	40.118	148.624	-38.473	40.117	148.623	-38.474
	150	-13.796	52.587	-38.473	-13.797	52.586	-38.474
(L)	120	-12.469	-57.540	-38.473	-12.469	-57.540	-38.474
(R)	120	-12.469	-57.540	11.527	-12.470	-57.539	11.526
	90	10.250	-27.249	11.527	10.250	-27.249	11.526
	60	14.780	10.343	11.527	14.780	10.343	11.526
	30	-0.093	45.163	11.527	-0.093	45.163	11.526
(L)	0	-30.383	67.883	11.527	-30.383	67.882	11.526
(R)	0	0.000	0.000	0.000	0.000	0.000	0.000



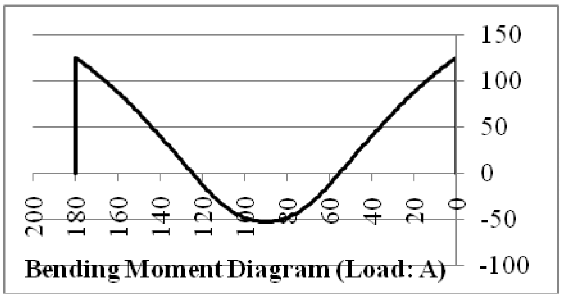
Twisting Moment Diagram



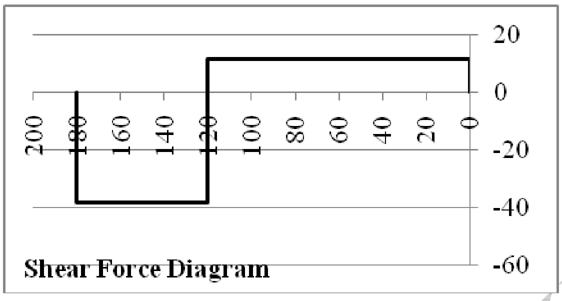
Twisting Moment Diagram (Load: A)



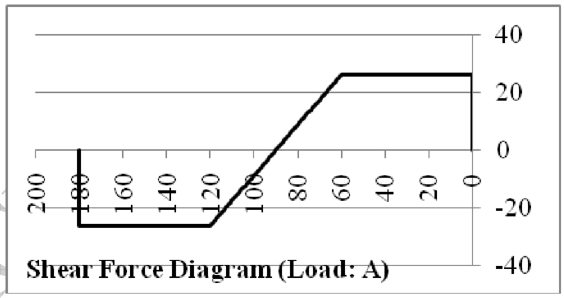
Bending Moment Diagram



Bending Moment Diagram (Load: A)



Shear Force Diagram



Shear Force Diagram (Load: A)

Example – IV: Cross-section – 0.300 m × 0.450 m, r = 5 m, φ = 180^o, w_1 = 10kN/m, w_2 = 10kN/m, α_1 = 60^o, α_2 = 120^o (Load: A)

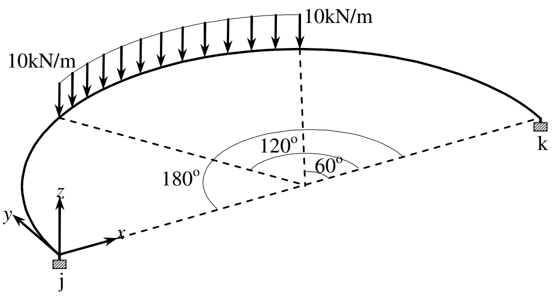


Fig. 5: Grid member subjected to UDL

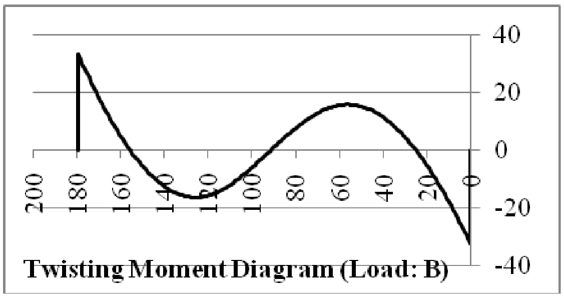
Table 4: Comparison of results of Example – IV

Section		Results of the Program			Results from STAAD.Pro		
Left/ Right	θ (°)	T_x (kNm)	M_x (kNm)	V_x (kN)	T_x (kNm)	M_x (kNm)	V_x (kN)
(L)	180	0.000	0.000	0.000	0.000	0.000	0.000
(R)	180	43.914	125.000	-26.180	43.913	125.000	-26.180
	150	-6.933	64.760	-26.180	-6.933	64.760	-26.180
	120	-20.847	-12.832	-26.180	-20.847	-12.832	-26.180
	90	0.000	-53.493	0.000	-0.001	-53.492	0.000
	60	20.847	-12.832	26.180	20.847	-12.831	26.180
	30	6.933	64.760	26.180	6.933	64.760	26.180
(L)	0	-43.914	125.000	26.180	-43.913	125.000	26.180
(R)	0	0.000	0.000	0.000	0.000	0.000	0.000

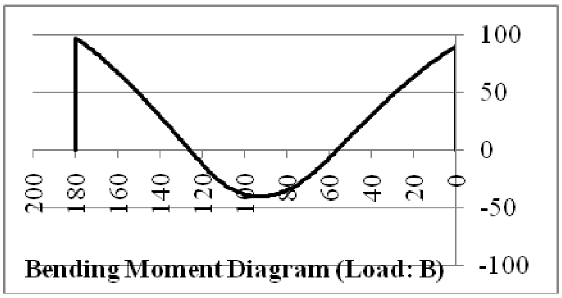
Example – V: Cross-section – 0.300 m × 0.450 m, r = 5 m, φ = 180^o, w_1 = 5kN/m, w_2 = 10kN/m, α_1 = 60^o, α_2 = 120^o (Load: B); and w_1 = 5kN/m, w_2 = 0kN/m, α_1 = 60^o, α_2 = 120^o (Load: C)

Table 5: Comparison of results of Example – V

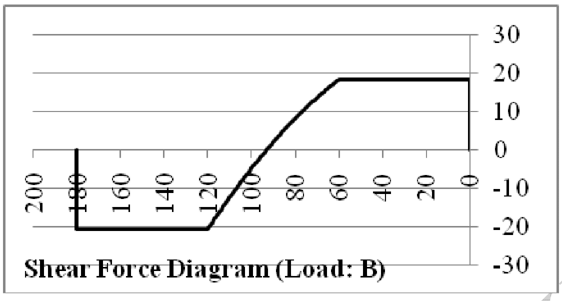
Section		Results of the Program (Load: B)			Results of the Program (Load: C)		
Left/ Right	θ (°)	T_x (kNm)	M_x (kNm)	V_x (kN)	T_x (kNm)	M_x (kNm)	V_x (kN)
(L)	180	0.000	0.000	0.000	0.000	0.000	0.000
(R)	180	33.406	97.507	-20.841	10.507	27.493	-5.339
	150	-5.862	49.045	-20.841	-1.070	15.715	-5.339
	120	-15.639	-12.558	-20.841	-5.208	-0.274	-5.339
	90	1.166	-40.119	2.067	-1.166	-13.373	-2.067
	60	15.631	-6.691	18.429	5.216	-6.142	7.751
	30	4.537	48.095	18.429	2.396	16.665	7.751
(L)	0	-32.464	89.993	18.429	-11.450	35.007	7.751
(R)	0	0.000	0.000	0.000	0.000	0.000	0.000
Section		Load: A = Load: B + C			Results from STAAD.Pro Load: A		
Left/ Right	θ (°)	T_x (kNm)	M_x (kNm)	V_x (kN)	T_x (kNm)	M_x (kNm)	V_x (kN)
(L)	180	0.000	0.000	0.000	0.000	0.000	0.000
(R)	180	43.913	125.000	-26.180	43.913	125.000	-26.180
	150	-6.932	64.760	-26.180	-6.933	64.760	-26.180
	120	-20.847	-12.832	-26.180	-20.847	-12.832	-26.180
	90	0.000	-53.492	0.000	-0.001	-53.492	0.000
	60	20.847	-12.833	26.180	20.847	-12.831	26.180
	30	6.933	64.760	26.180	6.933	64.760	26.180
(L)	0	-43.914	125.000	26.180	-43.913	125.000	26.180
(R)	0	0.000	0.000	0.000	0.000	0.000	0.000



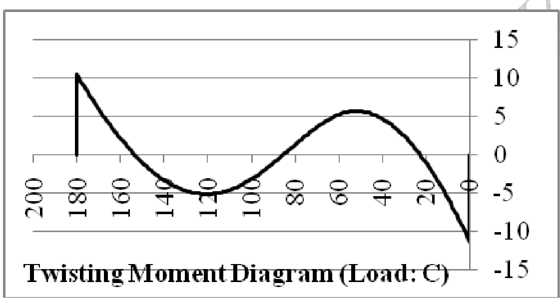
Twisting Moment Diagram (Load: B)



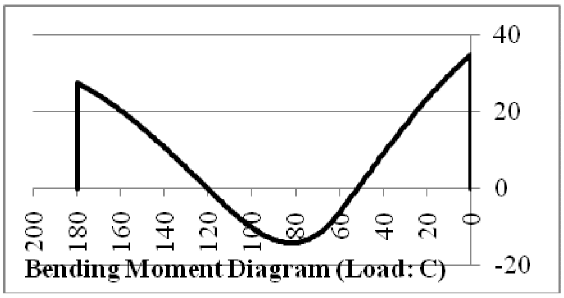
Bending Moment Diagram (Load: B)



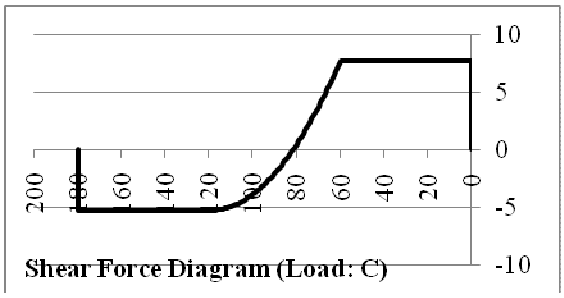
Shear Force Diagram (Load: B)



Twisting Moment Diagram (Load: C)



Bending Moment Diagram (Load: C)



Shear Force Diagram (Load: C)

CONCLUSION: The formulation of internal forces for the beam curved in plan is attuned for the computer programming. The results obtained from C++ program can be represented in the graphical form. The results, obtained using program, are validated by comparing results obtained through analysis software. The algebraic sum of results obtained for two complementary uniformly varying load cases are validated by analysis software. A comprehensive C++ program developed is giving accurate results of internal forces. The very small amount of difference is only due to truncation error. The formulation is applicable for most of practical load cases and its combination and fixed ends condition. The formulation can be extended for other support conditions.

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NOMENCLATURE:

E	:	Young's modulus of elasticity of material
G	:	Modulus of rigidity of material
I	:	Moment of inertia
J	:	Torsional moment of inertia
T_x	:	Twisting moment at any section x
M_x	:	Bending moment at any section x
ds	:	Small elemental length
T_k	:	Twisting moment at k-end
M_k	:	Bending moment at k-end
V_k	:	Vertical reaction at k-end
T_j	:	Twisting moment at j-end
M_j	:	Bending moment at j-end
V_j	:	Vertical reaction at j-end
$[GR]_k$	:	Strain energy due to the unit fixed end actions applied at k-end (flexibility coefficients of k-end)
$\{GL\}_k$	:	Strain energy due to the external loads
$[GR]_j$	:	Reactions at j-end due to the unit value of fixed end actions applied at k-end
$\{GL\}_j$	:	Reactions at j-end due to the external loads
$\{AR\}_k$	:	Fixed end reactions at k-end in member directions
$\{AR\}_j$	:	Fixed end reactions at j-end in member directions
r	:	Radius of segmental circular arch
ϕ	:	Angle between j-end and k-end
T	:	Twisting moment
M	:	Bending moment
P	:	Concentrated load
α	:	Angle between the k-end and point of application of external loads like concentrated load (P), bending moment (M) and twisting moment (T).
w_1, w_2	:	Beginning and terminating intensity of the uniformly varying load
α_1	:	Angle between the k-end and beginning point of the uniformly varying load
α_2	:	Angle between the k-end and termination point of the uniformly varying load
T_{xL}, M_{xL}, V_{xL}	:	Twisting moment, Bending moment and Shear force on left of the section at θ^o.
T_{xR}, M_{xR}, V_{xR}	:	Twisting moment, Bending moment and Shear force on right of the section at θ^o.